<u>Lemma</u>: $f: E \mapsto E'$, $g: E \mapsto E'$

f and g are continuous. Let $h: E \mapsto \mathbb{R}$ given by $h(x) = d(f(x), g(x))$. Then h is continuous.

~~proof~~: Let $x_0 \in E$, Let $\varepsilon > 0$

$h(x_0) = d(f(x_0), g(x_0))$

$h(x) = d(f(x), g(x))$

$\Rightarrow h(x) \le d(f(x), f(x_0)) + \underbrace{d(f(x_0), g(x_0))}_{= h(x_0)} + d(g(x_0), g(x))$

$\Rightarrow h(x) - h(x_0) \le d(f(x), f(x_0)) + d(g(x_0), g(x))$

similarly $h(x_0) - h(x) \le d(f(x_0), f(x)) + d(g(x), g(x_0))$

then $|h(x) - h(x_0)| \le d(f(x), f(x_0)) + d(g(x), g(x_0))$

Let $\delta_1 > 0$, s.t. $f(B_\delta(x_0)) \subset B_{\varepsilon/2}(f(x_0))$

Let $\delta_2 > 0$, s.t. $g(B_\delta(x_0)) \subset B_{\varepsilon/2}(g(x_0))$

Select $\delta = \min\{\delta_1, \delta_2\}$.

then $|h(x) - h(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, $h(x)$ is continuous.

<u>Definition</u>: E compact, E' metric

$$\mathcal{F} = \{f : E \mapsto E' : f \text{ is continuous}\}.$$

$$D(f, g) = \max_{x \in E} d(f(x), g(x)).$$

Let $h: E \mapsto \mathbb{R}$ and defined as.

$$h(x) = d(f(x), g(x)), \quad \forall x \in E.$$

By Lemma, h is continuous,

Since E is compact, $h(E)$ is compact.

So, h can attain its maximum.

Thus, $\boxed{D(f,g) \text{ is well defined.}}$

<u>Lemma</u>: D is a distance.

proof: (1) $\boxed{D(f,g) \geq 0, \ \forall f, g \in \mathcal{F}}$ ✓

(2) $D(f,g) = 0 \iff d(f(x), g(x)) = 0$ $\forall x \in E$, $\iff f(x) = g(x)$, $\forall x \in E \iff f = g$, i.e. $\boxed{D(f,g) = 0 \iff f = g}$

(3) $\boxed{D(f,g) = D(g,f)}$ ✓

(4) $f, g, h \in \mathcal{F}$

$$
\begin{aligned}
D(f,h) &= \max_{x \in E} d(f(x), h(x)) \\
&\leq \max_{x \in E} \left[d(f(x), g(x)) + d(g(x), h(x)) \right] \\
&\leq \max_{x \in E} d(f(x), g(x)) + \max_{x \in E} d(g(x), h(x)) \\
&= D(f,g) + D(g,h).
\end{aligned}
$$

i.e. $\boxed{D(f,h) \leq D(f,g) + D(g,h)}$

Proposition E is compact,

$f_n \to f$ in $(\mathcal{F}, D) \iff f_n \to f$ uniformly.

~~proof~~: $\Rightarrow$) Assume $f_n \to f$ in $(\mathcal{F}, D)$

Let $\varepsilon > 0$, $f_n \to f$ with distance D.

$\Leftrightarrow$ $\exists\, N > 0$, such that

$n > N \Rightarrow D(f_n, f) < \varepsilon$.

$\Rightarrow \max\limits_{x \in E} d(f_n(x), f(x)) < \varepsilon$.

Thus, $d(f_n(x), f(x)) < \varepsilon$, $\forall x \in E$.

i.e. $f_n \to f$ uniformly.

$\Leftarrow$) Assume $f_n \to f$ uniformly.

Let $\varepsilon > 0$, $\exists\, N > 0$ such that

$n > N \Rightarrow d(f_n(x), f(x)) < \varepsilon$, $\forall x \in E$

$\Rightarrow D(f_n, f) = \max\limits_{x \in E} d(f_n(x), f(x)) < \varepsilon$, $\forall n > N$

i.e. $f_n \to f$ in $(\mathcal{F}, D)$.

<u>Theorem</u> E compact and E' complete, then $(\mathcal{F}, D)$ is complete

~~proof~~: Let f_n be Cauchy in $(\mathcal{F}, D)$.

Let $\varepsilon > 0$, $\forall x \in E$

$d(f_m(x), f_n(x)) \leq D(f_m, f_n) < \varepsilon$

if $m, n \geq N$ for som $N > 0$.

By the proposition from last class,

$f_n \to f$ uniformly.

By the proposition in this class, $f_n \to f$.

in $(\mathcal{F}, D)$.

Problem 4.4

U, V are intervals in $\mathbb{R}$, $f: U \mapsto V$

onto and strictly increasing. Prove f and f^{-1}

are continuous.

~~proof~~ Let $\varepsilon > 0$, $x_0 \in U$.

Assume $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset V$

Since f is onto, $f^{-1}(f(x_0)-\varepsilon)$ and $f^{-1}(f(x_0)+\varepsilon) \in U$.

Since f is strictly increasing.

$f^{-1}(f(x_0)-\varepsilon) < x_0 < f^{-1}(f(x_0)+\varepsilon)$,

Let $\delta = \min\{f^{-1}(f(x_0)+\varepsilon - x_0,\ x_0 - f^{-1}(f(x_0)-\varepsilon)\}$.

then $(x_0-\delta, x_0+\delta) \subset (f^{-1}(f(x_0))-\varepsilon, f^{-1}(f(x_0))+\varepsilon)$

$$\Rightarrow f((x_0-\delta, x_0+\delta)) \subset f(f^{-1}(f(x_0))-\varepsilon, f^{-1}(f(x_0))+\varepsilon)$$

$$= (f(x_0)-\varepsilon, f(x_0)+\varepsilon) = B_\varepsilon(f(x_0)).$$

Thus, f is continuous.